THERMOCAPILLARY MOTION OF A LIQUID

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A number of works have been devoted to the exact or approximate theoretical treatment of thermocapillary convection. A one-dimensional model of the motion of a thin liquid layer with a linear temperature distribution on its free surface was discussed in [1]. The inconsistencies of the treatment of [1] were pointed out in [2]. Taking into account the comments of [2], the approximate solution of [1] was generalized in [3]. A closed form solution of the equations of free convection for plane-parallel steady flow in a horizontal liquid layer with a constant temperature gradient along the layer was worked out in [4]. Thermocapillary convection was considered, along with ordinary thermal convection. The solution of [4]* was extended in [5, 6] to different boundary conditions for the temperature and in [7] to the case of two immiscible liquid layers. The flow of two immiscible liquid layers in a plane inclined channel with nonisothermal parallel walls was considered in [8]. The effects of a pressure gradient, gravity, and capillary forces on the plane boundary between the two liquids were taken into account. Two-dimensional convection in a plane rectangular cell was studied numerically in [9] and in several later papers. When the motion is weak the problem can be solved analytically [10]. An asymptotic analysis was carried out in [11] for a shallow channel and low-intensity convection. The works cited above assumed that the liquid is heated from the bottom or through the sides of the container. Steady convection for the case of localized heating of a horizontal liquid layer from above was stimulated numerically in [12], and the development of convection was considered in [13]. Equations describing unsteady thermocapillary convection in thin liquid layers were derived in [14] and applied to certain problems. Exact solutions for the steady motion of a liquid in a half space subjected to nonuniform heating of its free surface were obtained in [15]. Spatially periodic convection induced by periodic transfer of heat from the bulk into the liquid layer was considered in the linear approximation in [16]. Heating in a transparent liquid can be accomplished by means of absorption of laser radiation. Periodic thermocapillary flow has been studied experimentally in [17].

We consider several other examples of flow due to the thermocapillary effect.

1. Couette Flow. Consider a horizontal layer consisting of light ($0 < y < h_1$) and heavy ($-h_2 < y < 0$) immiscible liquids (liquids 1 and 2, respectively). The pressure of the gas in the region above the layer is assumed to be constant. The free surface of the liquid $y = h_1$ is characterized by the surface tension α_1. The boundary between the two liquids $y = 0$ has the surface tension α_2. The lower boundary of the layer of the second liquid is a solid surface. Certain kinematic and dynamical conditions must be satisfied on the deformable boundaries. The dynamical boundary conditions couple the components of the stress tensor [18]. Although the densities $\rho_{1,2}$ and the transport coefficients of the liquids are assumed to be independent of temperature, the thermal and dynamical problems are coupled by the boundary conditions.

It can be shown that the boundaries of the liquids remain planes and a pressure gradient in the horizontal direction is not induced by the thermocapillary motion. The Navier–Stokes equation and the energy equation with the appropriate boundary conditions yield the following exact solution with a piecewise-linear velocity profile:

$$u_1 = u_0 + \frac{\alpha_1'(\beta + \gamma_1 h_1)}{\rho_1 \nu_1} y, \quad u_2 = u_0\left(1 + \frac{y}{h_2}\right),$$

*A critical discussion of [4] is contained in [13].

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$$u_0 = \frac{(\alpha_1' + \alpha_2')\,\beta h_2}{\rho_2 \nu_2} + \frac{\alpha_1' \gamma_1 h_1 h_2}{\rho_2 \nu_2}, \quad T_{1,2} = -x(\beta + \gamma_{1,2} y) + \theta_{1,2}(y), \tag{1.1}$$

$$\chi_{1,2}\theta_{1,2}'' + (\beta + \gamma_{1,2} y)\, u_{1,2} + \frac{\nu_{1,2}}{c_{1,2}} u_{1,2}'^2 = 0,$$

$$\theta_1(0) = \theta_2(0), \quad \varkappa_1 \theta_1'(0) = \varkappa_2 \theta_2'(0).$$

Here $u_{1,2}$ are the horizontal components of the velocities; $\alpha_{1,2}' = -d\alpha_{1,2}/dT$; β is the temperature gradient along the surface between the two liquids; γ_1, γ_2 are constants related by the equation $\varkappa_1\gamma_1 = \varkappa_2\gamma_2$; $\varkappa_{1,2}$ are the thermal conductivities of the two liquids; $\chi_{1,2}$ are the thermal diffusivities; $c_{1,2}$ are the heat capacities; $\nu_{1,2}$ are the kinematic viscosities; $u_0 = u(0)$ is the velocity on the boundary between the two liquids. The solution (1.1) for the temperature is identical to that found in [8]; when $\gamma_{1,2} = 0$ this solution reduces to that found in [4]. According to (1.1), the flow in the two layers is coupled and the velocity field does not depend on the form of the functions $\theta_{1,2}$.

Different values of the constants β and $\gamma_{1,2}$ correspond to different heat transfer problems. Suppose, for example, that the temperature of the free surface of the first liquid is constant $\beta + \gamma_1 h_1 = 0$. Then the velocity in the upper layer is constant ($u_1 = u_0$) and the solution does not depend on α_1' and ν_1. The flow is induced by capillary forces which arise because the surface between the two liquids is not isothermal.

The inverse effect is also possible where the upper layer flows with a linear velocity profile, while in the lower layer the velocity is constant and equal to zero, in view of the boundary condition on the tangential components of the velocity at the solid surface. In this case $(\alpha_1' + \alpha_2')\beta + \alpha_1'\gamma_1 h_1 = 0$ and capillary forces on the boundaries of the liquids are in opposite directions. The flow regimes in a viscous liquid layer of finite thickness with constant velocity are possibly of interest in applications. The motion is in the direction of decreasing temperature and upon solidification of such a flow inhomogeneities and local stresses in the solid phase will be reduced.

If $\beta = 0$ then the dependence of the solution on α_2' drops out. The motion is initiated by capillary forces on the free surface and the second liquid is drawn into the motion by viscous stresses. If $\beta = \gamma_2 h_2$, then the bottom of the tank is isothermal and the motion is a superposition of the flows generated by the two constant shear stresses on the moving boundaries of the layers.

In the case when the bottom of the tank $y = -h_2$ is inclined by an angle φ to the horizontal, the form of the solution (1.1) changes. The temperature field is given by the previous equations, but quadratic terms in the coordinates appear in the expressions for the velocities. These additional terms are due to the presence of body forces:

$$u_1 = u_0 + \frac{\alpha_1'(\beta + \gamma_1 h_1)}{\rho_1 \nu_1} y + \frac{g h_1^2 \sin\varphi}{2\nu_1} \frac{y}{h_1}\left(2 - \frac{y}{h_1}\right),$$
$$u_2 = \frac{g h_2^2 \sin\varphi}{2\nu_2}\left(1 - \frac{y^2}{h_2^2}\right) + \frac{y + h_2}{\nu_2 \rho_2}\left[\rho_1 h_1 g \sin\varphi + \alpha_1'(\beta + \gamma_1 h_1) + \alpha_2'\beta\right] \tag{1.2}$$

(g is the acceleration of gravity, and the velocity u_0 on the boundary between the two liquids is found from the second equation. The conditions under which the capillary force plays the dominant role are apparent from (1.2). The solution (1.2) is of interest in connection with the Flout process in the manufacture of glass, as noted in [7]. In this case there is a leakage of the glass layer on a layer of molten tin.

2. Steady Spatially Periodic Convection. A horizontal layer of incompressible liquid is bounded from below by the bottom of the tank $y = -h$ and from above by a free surface at $y = 0$. In the absence of thermal disturbances the liquid is at rest, has a constant temperature, and the equilibrium pressure distribution $p_0 = -\rho g y + \text{const}$.

We consider plane steady-state thermocapillary motion in the layer induced by a spatially periodic temperature disturbance $T = \Delta T \cos kx$ on the free surface of the liquid. As we shall see the motion differs from that studied in [16], where the convection was induced by a spatial modulation of heat transfer from the bulk. A surface thermal disturbance takes place, for example, in the interaction of laser radiation with an opaque liquid.

We linearize the equations of motion and the energy equation about the initial equilibrium state. Then it is possible to obtain simple analytical solutions. The boundary conditions are formulated assuming that the surface of the liquid is not deformed by thermocapillary motion. This means that only tangential stresses on the surface are taken into account, while capillary pressure and other effects associated with the curvature of the surface are neglected. However the pressure gradients induced by the motion of the liquid are directly related to the curvature of the free surface [2]. Therefore the calculation of the pressure field in the approximation of a plane surface is of little interest and it is necessary to eliminate the pressure gradient from the equations of motion. Then the components of the velocity satisfy the biharmonic equation, while the temperature disturbance T(x, y) and the vorticity Ω(x, y) satisfy Laplace's equation.

A solution of the problem with period 2π/k along the x axis can be constructed with nonlinear convective terms taken into account using regular perturbation theory with the perturbation ΔT. This was done in [19] in a treatment of thermogravitational convection. However, this is not convenient for our purposes. The solution given below, which is valid when the amplitude of the temperature modulation on the boundary of the layer is sufficiently small, is the first term of the perturbation expansion:

$$
\begin{gathered}
T=\theta(y)\cos kx,\ u=f'(y)\sin kx,\ v=-kf(y)\cos kx,\\
\Omega=F(y)\sin kx,\ F=k^2f-f'',\quad (d^2/dy^2-k^2)^2f=0,\quad (d^2/dy^2-k^2)\theta=0,\\
f(0)=f(-h)=f'(-h)=0,\ f''(0)=\alpha'\Delta Tk/(\nu\rho),\\
\theta(0)=\Delta T,\ \theta(-h)=0.
\end{gathered}
\tag{2.1}
$$

Here u is the horizontal component of the velocity and v is the vertical component. It is assumed that the temperature disturbance vanishes on the lower boundary of the layer. According to (2.1), the streamlines are determined by the equation $|f\sin kx|=C$. The motion of the liquid is a circulation along closed trajectories inside separate cells of length Δx = π/k. The solution of (2.1) is

$$
\begin{gathered}
f=\frac{\alpha'\Delta T}{\nu\rho a}[y\,\mathrm{sh}\,kh\,\mathrm{sh}\,(kh+ky)-kh\,(y+h)\,\mathrm{sh}\,ky],\\
\theta=\Delta T\,\frac{\mathrm{sh}(kh+ky)}{\mathrm{sh}\,kh},\quad a=\mathrm{sh}\,2kh-2kh,\\
F=\frac{2\alpha'\Delta Tk}{\nu\rho a}\,[kh\,\mathrm{ch}\,ky-\mathrm{sh}\,kh\,\mathrm{ch}\,(kh+ky)].
\end{gathered}
\tag{2.2}
$$

According to (2.2), F(−h) ≠ 0 if h is finite. In the limit h → ∞ the solution takes on the particularly simple form

$$
\theta=\Delta Te^{ky},\ F=-\frac{\alpha'\Delta Tk}{\nu\rho}e^{ky},\ f=\frac{\alpha'\Delta T}{2\nu\rho}ye^{ky}.
\tag{2.3}
$$

3. Oscillatory Thermocapillary Motion. We consider an example of plane flow depending explicitly on the time. Let an imcompressible liquid occupy the half space y < 0. We consider oscillatory thermocapillary motion caused by a periodic temperature disturbance T = ΔT cos kx × cos ωt on the free surface of the liquid. The problem is solved in the linear approximation, as in Sec. 2. We consider only the thermocapillary mechanism of convection and neglect buoyancy forces. This question will be discussed after obtaining the solution. In constructing the boundary conditions we assume that the surface of the liquid is undeformable due to capillary motion. This eliminates from the problem the excitation of surface waves and (in view of the incompresibility condition) internal waves. The disturbances of all quanitities must vanish in the limit y → −∞. We assume a solution of the form (2.1) where θ, f, and F are now functions of t and y. The functions T(x, y, t) and Ω(x, y, t) satisfy the heat conduction equation. The solution simplifies considerably because in a semi-infinite region a second boundary condition is known for Ω in the limit y → −∞. Therefore we can first obtain the solution for Ω and then, after using the relation (2.1) between f and F, find f(y, t) from a second-order equation. We then obtain the expressions

$$\theta = \Delta T e^{\delta y} \cos(my + \omega t), \quad F = -\frac{\alpha' \Delta T k}{\nu\rho} e^{\delta_1 y} \cos(m_1 y + \omega t),$$
$$f = \frac{\alpha' \Delta T k}{\rho\omega} [e^{\delta_1 y} \sin(m_1 y + \omega t) - e^{ky} \sin \omega t],$$
$$2\delta^2 = (k^4 + \omega^2/\chi^2)^{1/2} + k^2, \quad 2m^2 = (k^4 + \omega^2/\chi^2)^{1/2} - k^2,$$
$$2\delta_1^2 = (k^4 + \omega^2/\nu^2)^{1/2} + k^2, \quad 2m_1^2 = (k^4 + \omega^2/\nu^2)^{1/2} - k^2. \tag{3.1}$$

The solution (3.1) reduces to (2.3) when $\omega \to 0$. It follows from (3.1) that the oscillations of the different quantities are shifted in phase with respect to one another, and the phase shift depends on the y coordinate.

If instead of the temperature we specify the heat flux density q on the surface of the liquid, and we assume that it is described by the same equation as above, then the solution differs from (3.1) by the phase shift $\Delta\varphi = \arctan(m/\delta)$ and by the substitution $\Delta T \to \Delta q/\varkappa(k^4 + \omega^2/\chi^2)^{1/4}$. Therefore temperature oscillations with the same amplitude in different liquids correspond to different values of the heat flux. The amplitude of the velocity oscillations is proportional to the amplitude of the thermal disturbance and, like the phase shifts, depends on the thermal properties of the liquid.

The length scale in the horizontal direction is specified by the boundary condition. The solution (3.1) describes disturbances which damp out exponentially with depth into the liquid and resemble the solution of the well-known Stokes problem [18]. The penetration depths for the different quantities are in general different. For the velocity components the characteristic vertical length scale (since $\delta_1 > k$) is $\sim k^{-1}$, as in the horizontal direction. For large k, when δ, $\delta_1 \simeq k$, the penetration depths for temperature and curl of the velocity are also quantities $\sim k^{-1}$. But for small k, when $\delta \sim (\omega/\chi)^{1/2} \gg k$, $\delta_1 \sim (\omega/\nu)^{1/2} \gg k$, the penetration depths are small in comparison with k^{-1}.

It is known (see [4], for example) that thermogravitational convection can be neglected if the thickness of the liquid layer does not exceed a critical value h_*. Hence when $kh_* \gg 1$ the thermocapillary mechanism of convection will play the dominant role.

It is not difficult to write down conditions for the applicability of our treatment. When

$$\omega^2 \gg \frac{\alpha' \Delta T k^3}{\rho}, \quad \omega \gg \frac{\alpha' \Delta T k}{\nu\rho} \begin{cases} 1, & \mathrm{Pr} \leqslant 1, \\ \mathrm{Pr}, & \mathrm{Pr} > 1 \end{cases}$$

the neglected nonlinear terms will be small in comparison to the time derivatives or diffusion terms, respectively. Here Pr is the Prandtl number. According to (3.1), the response of the system to an oscillatory disturbance on the boundary is nonresonant. This is due to the assumed "rigidity" of the free surface in the transverse direction. In actuality, heat oscillations on the surface of the liquid will excite waves and in some cases lead to large-amplitude wave motion. Therefore our approach is applicable only when we are far from resonance.

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